

On the Hilbert function of the tangent cone of a monomial curve

M. D'Anna ^{*} M. Di Marca [†] V. Micale ^{*}

Abstract

In this paper we study the Hilbert function of $\mathrm{gr}_{\mathfrak{m}}(R)$, when R is a numerical semigroup ring or, equivalently, the coordinate ring of a monomial curve. In particular, we prove a sufficient condition for a numerical semigroup ring in order get a non-decreasing Hilbert function, without making any assumption on its embedding dimension; moreover, we show how this new condition allows to improve known results about this problem. To this aim we use certain invariants of the semigroup, with particular regard to its Apéry-set.

Keywords: Numerical semigroup, monomial curve, Hilbert function, Apéry set.

2010 MSC: 13A30; 13D40; 13H10.

Introduction

Let (R, \mathfrak{m}) be a Noetherian local ring with $|R/\mathfrak{m}| = \infty$ and let $\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$ be the associated graded ring of R with respect to \mathfrak{m} . The study of the properties of $\mathrm{gr}_{\mathfrak{m}}(R)$ is a classical subject in local algebra, not only in the general d -dimensional case, but also under particular hypotheses (that allow to obtain more precise results). A classical problem in this context is to study the Hilbert function of R , i.e., by definition, the Hilbert function of $\mathrm{gr}_{\mathfrak{m}}(R)$.

^{*}Università di Catania, Dipartimento di Matematica e Informatica, Viale A. Doria, 6, 95125 Catania, Italy

[†]Scuola Superiore di Catania, Via Valdisavoia, 9, 95123 Catania, Italy

In this paper we are interested in the Hilbert function of R , when R is a numerical semigroup ring. The study of numerical semigroup rings is motivated by their connection to singularities of monomial curves and by the possibility of translating algebraic properties into numerical properties (see e.g. [2]). However, even in this particular case, many pathologies occur, hence these rings are also a great source of interesting examples.

From the geometrical point of view, given a numerical semigroup S generated by n coprime integers g_1, g_2, \dots, g_n , the numerical semigroup ring, $R = k[[S]]$ is the completion of the local ring at the origin of the monomial curve $C = C(g_1, \dots, g_n)$ parameterized by $x_1 = t^{g_1}, \dots, x_n = t^{g_n}$. Hence its associated graded ring is the coordinate ring of the tangent cone of C in the origin. Moreover, $\text{gr}_{\mathfrak{m}}(R)$ is isomorphic to the ring $k[x_1, x_2, \dots, x_n]/I(C)_*$, where $I(C)$ is the defining ideal of C and $I(C)_*$ is the ideal generated by the homogeneous terms of least degree of the polynomials in $I(C)$.

One classical problem about the Hilbert function is to find conditions on R or $\text{gr}_{\mathfrak{m}}(R)$ to get a non decreasing function. In the context of one-dimensional local rings, it is well known that the Hilbert function is not decreasing if the embedding dimension is at most 3 (see [6] and [7]) and counterexamples (for reduced one-dimensional rings) were given for embedding dimension bigger than or equal to 4 (see [15] and [9]). However, for semigroup rings, there are no examples of decreasing Hilbert function when the embedding dimension is smaller than 10; so the problem is still open for semigroup rings R with embedding dimension $4 \leq e.d.(R) \leq 9$.

Another open problem in the one-dimensional case was posed by Rossi in [17]: if R is a Gorenstein one-dimensional local ring, is it true that the Hilbert function of $\text{gr}_{\mathfrak{m}}(R)$ is not decreasing? In the context of semigroup rings, it is equivalent to ask whether the Hilbert function of $k[[S]]$, with S symmetric, is non-decreasing.

The problem if the Hilbert function of a semigroup ring is non-decreasing, has been extensively studied. If $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, then the problem becomes trivial thanks to a result of A. Garcia in [8]. For the general case, recent results can be found, e.g., in [1], where many families of non-decreasing Hilbert function of semigroup rings are obtained by using the technique of gluing semigroups (see also [11]), and in [16], where the authors study particular 4-generated semigroups. Furthermore, in [4] new results on this problem are obtained introducing the Apéry-table of a semigroup; in particular, the authors proved that if S is 4-generated and if the tangent

cone is Buchsbaum, then $k[[S]]$ has non-decreasing Hilbert function.

The main result of this paper is a sufficient condition for a numerical semigroup ring in order to get a non-decreasing Hilbert function, without making any assumption on its embedding dimension (see Theorem 2.3 and Corollary 2.4); to this aim we use certain invariants of the semigroup, with particular regard to its Apéry-set. Successively, a careful use of the proof of the main result allows us to get a computationally more efficient, necessary condition for the decreasing of the Hilbert function (see Proposition 2.9 and the subsequent remark). Finally, we show how these results can be applied to improve known results about this problem (see Corollaries 2.8, 2.12 and 2.13).

1 Preliminaries

We start this section recalling some well known facts on numerical semigroups and semigroup rings. For more details see, e.g., [2].

A *numerical semigroup* S is a subsemigroup of $(\mathbb{N}, +)$ that includes 0. There is a natural partial order on S that is defined as follows: let $a, b \in S$, then

$$a \leq_S b \iff \exists u \in S : a + u = b.$$

The set of minimal elements with respect to this order is called *minimal set of generators* of S . It is always finite because, by definition, for any $s \in S, s \neq 0$, two minimal generators have to be different modulo s . Once fixed the minimal set of generators, each element of S can be written as finite sum of these elements. Hence S is determined by its minimal set of generators. We denote by $\langle g_1, g_2, \dots, g_n \rangle$ the numerical semigroup S whose minimal set of generators is $\{g_1, g_2, \dots, g_n\}$, where $g_1 < g_2 < \dots < g_n$. Since the semigroup $S = \langle g_1, g_2, \dots, g_n \rangle$ is isomorphic to $\langle dg_1, dg_2, \dots, dg_n \rangle$ for any $d \in \mathbb{N} \setminus 0$, we can assume that $\gcd(g_1, g_2, \dots, g_n) = 1$. This is equivalent to say that $|\mathbb{N} \setminus S|$ is finite, so it is well defined the maximum of the numbers that does not belong to the semigroup, called *Frobenius number of S* and denoted by f .

From now on, we will call a numerical semigroup simply semigroup.

A *relative ideal* of a semigroup S is a set $H \subset \mathbb{Z}$, $H \neq \emptyset$, such that $H + S \subseteq H$ and $H + s \subseteq S$, for some $s \in S$; if $H \subseteq S$, it is called *ideal*. If H and L are relative ideals, then also $kH = \{h_1 + h_2 + \dots + h_k : h_1, h_2, \dots, h_k \in$

$H\}$ (for all $k \in \mathbb{N}$) and $H -_{\mathbb{Z}} L = \{z \in \mathbb{Z} : z + l \in H, \forall l \in L\}$ are relative ideals. The ideal $M = S \setminus \{0\}$ is called *maximal ideal*.

Let k be an infinite field and let $S = \langle g_1, g_2, \dots, g_n \rangle$; the ring $R = k[[t^S]] = k[[t^{g_1}, t^{g_2}, \dots, t^{g_n}]]$ is called *semigroup ring associated to S* . The ring R is a one-dimensional local domain, with maximal ideal $\mathfrak{m} = (t^{g_1}, t^{g_2}, \dots, t^{g_n})$ and quotient field $k((t))$. Considering the \mathfrak{m} -adic filtration, let $gr^h(R)$ be the quotient $\mathfrak{m}^h/\mathfrak{m}^{h+1}$. From the direct sum of the $gr^h(R)$ we obtain the *associated graded ring* $gr_{\mathfrak{m}}(R)$ explicitly defined as $gr_{\mathfrak{m}}(R) = \bigoplus_{h \geq 0} \mathfrak{m}^h/\mathfrak{m}^{h+1}$. Setting $k = R/\mathfrak{m}$, the *Hilbert function* of R is then given by

$$H_R(h) = \dim_k gr^h(R), \quad \forall h \in \mathbb{N}.$$

There exists a strong connection between a semigroup and its associated ring. In fact, through the *natural valuation* function $v : k((t)) \rightarrow \mathbb{Z} \cup \infty$, that is

$$v\left(\sum_{n=i}^{\infty} r_n t^n\right) = i, \quad i \in \mathbb{Z}, \quad r_i \neq 0,$$

we get $v(R) = S$ and many other properties. For example, if I and J are fractional ideal of R , then $v(I)$ and $v(J)$ are relative ideal of S and so are $v(I \cap J)$, $v(I : J)$ and $v(I^n)$ for all $n \in \mathbb{N}$. Furthermore, if I and J are monomial fractional ideals, the following relations hold: $v(I \cap J) = v(I) \cap v(J)$, $v(I : J) = v(I) - v(J)$ and $v(I^n) = nv(I)$. Moreover, if $J \subseteq I$ are fractional ideals of R , then:

$$\dim_k(I/J) = |v(I) \setminus v(J)|.$$

These facts hold, in particular, for $I = \mathfrak{m}^h$ and $J = \mathfrak{m}^{h+1}$, for all $h \geq 0$. Therefore, since $v(\mathfrak{m}) = M$, the Hilbert function of the semigroup ring H_R is equivalent to the *Hilbert function of S* which is

$$H_S(h) = |hM \setminus (h+1)M|, \quad \forall h \in \mathbb{N}$$

(when $h = 0$ we set, as usual, $\mathfrak{m}^0 = R$ and $0M = S$).

We denote by $\text{Ap}(S)$ the Apéry-set of S with respect to the smallest generator g_1 , which is the set $\{\omega_0, \omega_1, \dots, \omega_{g_1-1}\}$, where $\omega_i = \min\{s \in S : s \equiv i \pmod{g_1}\}$.

The numerical semigroup $S' = \bigcup_i (iM -_{\mathbb{Z}} iM)$ is called *blow up of S* and it corresponds to the blow up of R . By [13, Lemma 1], S is generated

by $\{g_1, g_2 - g_1, \dots, g_n - g_1\}$ (but this is not necessarily its minimal set of generators). Moreover $S' \supseteq hM - hg_1 = \{s - hg_1 : s \in hM\}$ (for every $h \geq 1$) and the equality holds for every h large enough. The Apéry-set of S' with respect to g_1 is denoted by $\text{Ap}(S') = \{\omega'_0, \omega'_1, \dots, \omega'_{g_1-1}\}$.

We recall two important sets of invariants of S , introduced by Barucci and Fröberg in [3]. For each $i = 0, 1, \dots, g_1 - 1$, let a_i be the only integer such that $\omega'_i + a_i g_1 = \omega_i$ and let $b_i = \max\{l : \omega_i \in lM\}$. Clearly $b_0 = a_0 = 0$. Furthermore, Barucci and Fröberg proved that $1 \leq b_i \leq a_i$ for all i (see [3], Lemma 2.4) and that $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay if and only if the equality $a_i = b_i$ holds for each i (see [3], Theorem 2.6).

We will need also to consider another set of invariants introduced in [5] and related to the previous ones: $c_i = \min\{h : \omega'_i \in hM - hg_1\}$. We have that $a_i \leq c_i$ (for every i) and that $b_i < a_i$ if and only if $a_i < c_i$ (see [5], Proposition 3.5).

2 Relation between Apéry-set and Hilbert function

In this section we relate the coefficient a_i and b_i introduced in the previous section with the value of the Hilbert function.

From the characterization of H_R given in the first section, it is obvious that H_R is non-decreasing if and only if

$$|(h-1)M \setminus hM| \leq |hM \setminus (h+1)M|, \quad \forall h \geq 1.$$

In order to study this inequality, it is natural to consider the elements s belonging to $(h-1)M \setminus hM$ and to add g_1 : now, if $s + g_1 \in hM \setminus (h+1)M$, for all $s \in (h-1)M \setminus hM$, we get an injective function between the two sets. If this is the case for every $h \geq 1$, then H_R is non-decreasing. We will see that this situation corresponds to the case $\text{gr}_{\mathfrak{m}}(R)$ Cohen-Macaulay. On the contrary, it can happen that the set

$$D_h := \{s \in (h-1)M \setminus hM : s + g_1 \in (h+1)M\}$$

is non empty for some $h \geq 2$.

Let r be the reduction number of \mathfrak{m} , that is the minimal natural number such that $\mathfrak{m}^{r+1} = x\mathfrak{m}^r$, with x a superficial element of R (recall that such number r exists by [14, Theorem 1, Section 2]). We notice that, in the

semigroup ring case, the valuation of x is necessarily $v(x) = g_1$; hence the multiplicity of R , i.e. $\dim_k(\mathfrak{m}^h/\mathfrak{m}^{h+1}) = |hM \setminus (h+1)M|$ (for any $h \geq r$), coincides with g_1 .

Let $s \in S$; the maximal index h such that $s \in hM$ is called the *order* of s and it is denoted by $\text{ord}(s)$. We note that $\text{ord}(\omega_i) = b_i$. Since $\text{ord}(s) = h$ if and only if $s \in hM \setminus (h+1)M$, we often say in this case that s is *on the h -th level*. We also say that an element s *skips the level when adding g_1* if $\text{ord}(s) = h$ and $\text{ord}(s + g_1) > h + 1$. With this terminology, we can say that D_h is given by the elements on the $(h-1)$ -th level that skip the level when adding g_1 .

Let $D = \cup_{i \geq 2} D_i$. Notice that the condition $D = \emptyset$ is equivalent to say that the image of t^{g_1} in $\text{gr}_{\mathfrak{m}}(R)$ is not a zero-divisor, i.e., by [8], $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay.

The following result shows that the condition $a_i > b_i$ for ω_i in $\text{Ap}(S)$ is related with the elements in D .

Proposition 2.1. *Let S be a semigroup. For each index i there exists an element $s \in D$, $s \equiv i \pmod{g_1}$ if and only if $a_i > b_i$. In particular, if $a_i = b_i$ for every i , then $D = \emptyset$.*

Proof. (\Rightarrow) Let $s \in (h-1)M \setminus hM$ and let $s \equiv i \pmod{g_1}$; hence $s = \omega_i + \lambda g_1$. We use the induction on λ to prove that $a_i = b_i$ implies $s \notin D_h$.

For $\lambda = 0$, we have $\omega_i \in b_i M \setminus (b_i + 1)M$; if we suppose that $\omega_i + g_1 \in (b_i + 2)M$ we get $\omega_i + g_1 = \omega'_i + a_i g_1 + g_1 = (\omega'_i - g_1) + (b_i + 2)g_1 \in (b_i + 2)M$, that implies $\omega'_i - g_1 \in S'$, against the minimality of ω'_i in S' .

We now suppose the thesis true for $\lambda - 1 \geq 0$ and we fix $s = \omega_i + \lambda g_1$. By the inductive hypothesis $s \in (b_i + \lambda)M \setminus (b_i + \lambda + 1)M$; again, if we assume $s + g_1 \in (b_i + \lambda + 2)M$ then we could write $s + g_1 = (\omega'_i - g_1) + (b_i + \lambda + 2)g_1 \in (b_i + \lambda + 2)M$, in contrast with the definition of ω'_i .

(\Leftarrow) We know that $a_i > b_i$ implies $c_i > a_i$. Let us consider the element $s' = \omega'_i + c_i g_1 = \omega'_i + a_i g_1 + (c_i - a_i)g_1$. By definition of c_i , we have $s' \in c_i M$. Moreover, $\omega_i = \omega'_i + a_i g_1 \in b_i M \setminus (b_i + 1)M$ and $s' = \omega_i + (c_i - a_i)g_1$. Hence the inequality $b_i + c_i - a_i < c_i$ implies that there must be an element $s = \omega_i + \lambda g_1$ (for some λ , $0 \leq \lambda \leq c_i - a_i - 1$) that skips the level when adding g_1 , i.e. $s \in D$. \square

In [3, Theorem 2.6], the authors prove (in the more general context of one-dimensional analytically irreducible rings) that $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay if

and only if $a_i = b_i$, for every $i = 0, \dots, g_1 - 1$. As we noticed above, the cohen-macaulayness of $\text{gr}_{\mathfrak{m}}(R)$ is equivalent to $D = \emptyset$; hence the previous proposition give a different proof of the result of Barucci and Froberg in the semigroup ring case. Moreover, as a corollary, we get the well known result that, if $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, then H_R is non-decreasing.

In order to compare $|(h-1)M \setminus hM|$ and $|hM \setminus (h+1)M|$, we want to take into account the number of the skipping elements in each level. Hence, for each $h \geq 1$, we set

$$C_h = \{s \in hM \setminus (h+1)M : s - g_1 \notin (h-1)M \setminus hM, \};$$

in other words, C_h is the set of elements on the h -th level which don't come from any element on the previous level by adding g_1 . This means that if $s \in C_h$, then either $s - g_1$ is a skipping element coming from a level lower than the $(h-1)$ -th level, or s is an element of $Ap(S)$ of order h .

We notice that the sets C_h and D_h arise naturally in this context and were already defined in [4]; hence we conformed our terminology to the names appearing in that paper.

With this notation it is straightforward that $|(h-1)M \setminus hM| \leq |hM \setminus (h+1)M|$ if the number of elements on the $(h-1)$ -th level that skip level when adding g_1 is smaller than or equal to the number of elements in C_h , i.e. $|D_h| \leq |C_h|$. Hence

$$H_R \text{ is non-decreasing} \iff |D_h| \leq |C_h|, \quad \forall h \in \{2, \dots, r\}$$

(where r is the reduction number of \mathfrak{m} ; notice also that for $h = 1$ it is always true that $|S \setminus M| = 1 \leq |M \setminus 2M|$ or, equivalently, $D_1 = \emptyset$).

Our next goal is to find conditions for determining an injective function from D_h to C_h . We recall that each element s in the semigroup can be written as linear combination of the generators with coefficient in \mathbb{N} . If we have

$$s = \lambda_1 g_1 + \lambda_2 g_2 + \dots + \lambda_n g_n, \quad \lambda_i \in \mathbb{N},$$

we say that this is a *maximal representation* of s if $\sum_{i=1}^n \lambda_i = \text{ord}(s)$. The maximal representation of an element is not unique in general. If we have two maximal representations of the same element s , we will write

$$\lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_n g_n \prec \lambda'_1 g_1 + \lambda'_2 g_2 + \cdots + \lambda'_n g_n$$

if $(\lambda_1, \lambda_2, \dots, \lambda_n)$ is smaller than $(\lambda'_1, \lambda'_2, \dots, \lambda'_n)$ in the usual lexicographic order in \mathbb{N}^n .

The following lemma is similar to [4, Lemma 4.2(2)]. We prove it in this form, since we will need it for the proof of the main theorem.

Lemma 2.2. *For every index $h \geq 2$ there exists a function $\psi : D_h \rightarrow C_h$.*

Proof. Let $s \in D_h$. Then $\text{ord}(s) = h - 1$ and $s + g_1 \in (h + 1)M$; hence there exists $k \geq 2$ such that $s + g_1 \in (h + k - 1)M \setminus (h + k)M$. Let

$$s + g_1 = g_{l_1} + g_{l_2} + \cdots + g_{l_h} + g_{l_{h+1}} + \cdots + g_{l_{h+k-1}},$$

(with $g_{l_1} \leq g_{l_2} \leq \cdots \leq g_{l_{h+k-1}}$) be the greatest among all the maximal representations of $s + g_1$ with respect to the Lex order.

We define $\psi(s) := g_{l_1} + g_{l_2} + \cdots + g_{l_h}$; hence we have $\psi(s) \in hM \setminus (h + 1)M$, because it is part of a maximal representation. Let $s' := \psi(s) - g_1$ and let us assume, by contradiction, that $s' \in (h - 1)M$. We get $s + g_1 = s' + g_1 + g_{l_{h+1}} + \cdots + g_{l_{h+k-1}}$ that implies $s = s' + g_{l_{h+1}} + \cdots + g_{l_{h+k-1}} \in hM$, against the fact that $\text{ord}(s) = h - 1$. Hence $\psi(s) - g_1 \notin (h - 1)M$ and $\psi(s) \in C_h$. \square

Now we give a sufficient condition for S in order to have $|(h - 1)M \setminus hM| \leq |hM \setminus (h + 1)M|$. The Example 2.5 below will illustrate the procedure of the proof of the next theorem.

Theorem 2.3. *If $|D_h| \leq h + 1$, then there exists an injective function $\tilde{\psi} : D_h \rightarrow C_h$.*

Proof. Let s be an element in D_h , and let

$$s + g_1 = g_{l_1} + g_{l_2} + \cdots + g_{l_h} + g_{l_{h+1}} + \cdots + g_{l_{h+k-1}},$$

(with $g_{l_1} \leq g_{l_2} \leq \cdots \leq g_{l_{h+k-1}}$) be the greatest among all the maximal representations of $s + g_1$ with respect to the Lex order.

As in Lemma 2.2 we map s to $\psi(s) = g_{l_1} + g_{l_2} + \cdots + g_{l_h}$. Let $J = |D_h|$ and let

$$\psi(D_h) = \{\psi(s_1), \psi(s_2), \dots, \psi(s_J)\}$$

without removing the possible repetitions. Hence $|\psi(D_h)| = |D_h| \leq h + 1$. Let us order $\psi(D_h)$ according to the decreasing Lex order:

$$\psi(D_h) = \{\psi_1, \psi_2, \dots, \psi_J\}, \quad \psi_1 \succeq \psi_2 \succeq \dots \succeq \psi_J,$$

where $\psi_j = \psi(s_m)$ for some $1 \leq m \leq J$.

If all the images $\psi(s_m)$ are pairwise different, we get the thesis with $\tilde{\psi} = \psi$. Otherwise, let a be the minimum among the indexes such that $\psi_a = \psi_{a+1}$ and let s_u and s_v be the pre-images of ψ_a and ψ_{a+1} , respectively.

Let $\psi_a = \psi_{a+1} = g_{l_1} + g_{l_2} + \dots + g_{l_h}$. Since $s_u \neq s_v$, there exists a generator g_p , with $g_p > g_{l_h}$, that appears in the maximal representations of $s_u + g_1$ (or $s_v + g_1$) and does not appear in $\psi_a = \psi_{a+1}$ (otherwise, if both $s_u + g_1$ and $s_v + g_1$ have maximal representations involving only g_{l_1}, \dots, g_{l_h} , by $\psi_a = \psi_{a+1}$ we would get $s_u + g_1 \leq_S s_v + g_1$, or viceversa, and consequently $s_u \leq_S s_v$, or viceversa; contradiction against the fact that s_u and s_v have both order $h - 1$). Without loss of generality we can assume that g_p appears in the representations of $s_v + g_1$.

Now we can define a new function ψ' on D_h so that $\psi'(s) = \psi(s)$, for every $s \in D_h$, $s \neq s_v$, and $\psi'(s_v) = \psi_{a+1} - g_{l_h} + g_p = g_{l_1} + g_{l_2} + \dots + g_{l_{h-1}} + g_p$.

We now have $\psi'(s_v) \prec \psi(s_u) = \psi_a$. The new set of images is

$$\psi'(D_h) = [\psi(D_h) \cup \{\psi'(s_v)\}] \setminus \{\psi_{a+1}\}.$$

We reorder $\psi'(D_h)$ and we rename the elements ψ'_j , for $j = a + 1, \dots, J$ according to the decreasing Lex order. We have:

$$\psi'_1 \succ \psi'_2 \succ \dots \succ \psi'_a \succeq \psi'_{a+1} \succeq \dots \succeq \psi'_J.$$

Again, if all the elements in $\psi'(D_h)$ are pairwise different, we get the thesis. Otherwise, we repeat the same argument as above by taking the minimum of the indices for which we have an equality in the chain and we redefine the correspondent images. We observe that this index could be a again. In this case, we are sure that $\psi_{a+1} \neq \psi'_{a+1}$ and we can compare the two pre-images of ψ'_a and ψ'_{a+1} as in the previous step. By redefining one of them, we get a new set of images $\psi''(D_h)$.

There is the possibility that by ordering $\psi''(D_h)$ we find again an equality for the index a . We note that this event can happen at most $J - a$ times and no conditions are required to redefine the function at each step.

After a finite number of steps (say $w \geq 1$) we will have:

$$\psi_1^{(w)} \succ \psi_2^{(w)} \succ \dots \succ \psi_a^{(w)} \succ \psi_{a+1}^{(w)} \succeq \psi_{a+2}^{(w)} \dots \succeq \psi_J^{(w)}.$$

When this condition is satisfied we say that we have performed the first block of steps.

Again, if the images of $\psi^{(w)}$ are all pairwise different, we get the thesis with $\psi = \psi^{(w)}$. Otherwise, let b be the minimum among the indexes such that $\psi_b^{(w)} = \psi_{b+1}^{(w)}$ (with $b > a$). Let again $s_u \neq s_v$ be the pre-images of $\psi_b^{(w)}$ and $\psi_{b+1}^{(w)}$, respectively; moreover, we set $\psi_b^{(w)} = \psi_{b+1}^{(w)} = g_{l_1} + g_{l_2} + \dots + g_{l_{h-1}} + g_{l_q}$ (with $g_{l_1} \leq g_{l_2} \leq \dots \leq g_{l_{h-1}} \leq g_{l_q}$). This time, since $s_u \neq s_v$, there exists a generator g_p , with $g_p > g_{l_{h-1}}$, that appears in the maximal representations of $s_u + g_1$ (or $s_v + g_1$) (otherwise, if both $s_u + g_1$ and $s_v + g_1$ have maximal representations involving only $g_{l_1}, \dots, g_{l_{h-1}}$, since $g_{l_1} \leq g_{l_2} \leq \dots \leq g_{l_{h-1}}$ are the first $h-1$ elements in the maximal representation of both $s_u + g_1$ and $s_v + g_1$, we would get $s_u + g_1 \leq_S s_v + g_1$, or viceversa, and consequently $s_u \leq_S s_v$, or viceversa; contradiction against the fact that s_u and s_v have both order $h-1$. Notice that in this case we are not able any more to compare g_p and g_{l_q}). Without loss of generality we can assume that g_p appears in the representations of $s_v + g_1$.

We can define a new function $\psi^{(w+1)}$ on D_h so that $\psi^{(w+1)}(s) = \psi^{(w)}(s)$, for every $s \in D_h$, $s \neq s_v$, and $\psi^{(w+1)}(s_v) = \psi_{b+1}^{(w)} - g_{l_{h-1}} + g_p = g_{l_1} + g_{l_2} + \dots + g_{l_{h-2}} + g_p + g_{l_q}$ (or $g_{l_1} + g_{l_2} + \dots + g_{l_{h-2}} + g_{l_q} + g_p$, if $g_p > g_{l_q}$). This means that at this and at all the subsequent steps we will rearrange the summands in non-decreasing order. As in the previous block of steps, we go on until we get a $\psi^{(z)}$ such that

$$\psi_1^{(z)} \succ \psi_2^{(z)} \succ \dots \succ \psi_a^{(z)} \succ \dots \succ \psi_b^{(z)} \succ \psi_{b+1}^{(z)} \succeq \psi_{b+2}^{(z)} \dots \succeq \psi_J^{(z)}.$$

When this condition is satisfied we say that we have performed the second block of steps.

We would like to proceed until we obtain a chain of proper inequalities; i.e., denoting the last defined function by ψ , until we get:

$$\widetilde{\psi_1} \succ \widetilde{\psi_2} \succ \dots \succ \widetilde{\psi_J}.$$

To this aim, in the worst case, we need to perform $J - a$ blocks of steps, where a is the index of the first equality. Since at the j -th block of steps we substitute $g_{l_{h-j+1}}$ with the new generator g_p , we are sure that we can perform

h blocks of steps. Hence, in order to get the desired injective function, it is sufficient that $J - a \leq h$. Since $a \geq 1$ and $J = |D_h|$ we get the thesis.

We finally observe that every $\widetilde{\psi_a}$ is still an element of C_h , coming from some $s \in D_h$. Furthermore, if $\widetilde{\psi_a} \succ \widetilde{\psi_b}$, then $\widetilde{\psi_a} \neq \widetilde{\psi_b}$, since we are assuming that the summands of $\widetilde{\psi_a}$ (for every index a) are in non-decreasing order. Hence $\widetilde{\psi}$ is an injective function. \square

Corollary 2.4. *If $|D_h| \leq h + 1$ for every $h \geq 2$, then H_R is non-decreasing.*

The next example is appropriate to illustrate the procedure of the proof of the main theorem.

Example 2.5. Let $S = \langle 24, 25, 36, 51, 54 \rangle$. Its Hilbert function is non-decreasing; in fact, it assumes the following values

$$1, 5, 11, 16, 19, 20, 21, 22, 22, 22, 22, 23, 24, \rightarrow .$$

We have $|D_2| = 1$, $|D_3| = 3$, $|D_4| = 4$, $|D_5| = 4$ and $|D_h| \leq 3$ for every $h \geq 5$. Hence it is fulfilled the condition of the previous theorem and corollary. Let us analyze $D_5 = \{s \in 4M \setminus 5M : s + 24 \in 6M\} = \{126, 137, 155, 166\}$. We have

$$\begin{aligned} 126 + 24 &= 6 \cdot 25 \\ 137 + 24 &= 5 \cdot 25 + 36 \\ 155 + 24 &= 5 \cdot 25 + 54 \\ 166 + 24 &= 4 \cdot 25 + 36 + 54 \end{aligned}$$

The function ψ defined in Lemma 2.2, gives

$$\psi(126) = \psi(137) = \psi(155) = 5 \cdot 25 \succ \psi(166) = 4 \cdot 25 + 36.$$

Following the proof of Theorem 2.3, we have $a = 1$ and we define $\psi'(137) = 4 \cdot 25 + 36$: now we have $\psi'(126) = \psi'(155) \succ \psi'(137) = \psi'(166)$. So again we have an equality for $a = 1$; we are forced to define $\psi''(155) = 4 \cdot 25 + 54$ and we get: $\psi''(126) \succ \psi''(137) = 4 \cdot 25 + 36 = \psi''(166) \succ \psi''(155)$. Now we have completed the first block of steps and we have an equality for $b = 2$.

In the maximal representation of $166 + 24$, it appears 54; hence we can define $\psi^{(3)}(166) = 3 \cdot 25 + 54 + 36$. After reordering its summands we obtain $\psi^{(3)}(126) = 5 \cdot 25 \succ \psi^{(3)}(137) = 4 \cdot 25 + 36 \succ \psi^{(3)}(155) = 4 \cdot 25 + 54 \succ \psi^{(3)}(166) = 3 \cdot 25 + 36 + 54$. Now we have completed the second block of steps and, since we have inequalities for every index of the chain, we can set $\widetilde{\psi} = \psi^{(3)}$.

Notice that $C_4 = \{125, 136, 154, 165, 191\} = \tilde{\psi}(D_4) \cup \{191\}$ and that $191 - 24 \notin S$.

Remark 2.6. We note that the result of Theorem 2.3 is the best possible as the semigroup $S = \langle 13, 19, 24, 44, 49, 54, 55, 59, 60, 66 \rangle$ has $|D_2| = 4$ and H_R decreasing (see [12]). More precisely we have:

$$44 + 13 = 19 + 19 + 19$$

$$49 + 13 = 19 + 19 + 24$$

$$54 + 13 = 19 + 24 + 24$$

$$59 + 13 = 24 + 24 + 24$$

The function ψ , defined in Lemma 2.2, gives $\psi(44) = \psi(49) = 19 + 19 \succ \psi(54) = 19 + 24 \succ \psi(59) = 24 + 24$. If we try to follow the procedure of Theorem 2.3, we have to define $\psi'(49) = 19 + 24$ (in this case the first block of steps consists of one step); now we have the equality $\psi'(49) = \psi'(54)$ and we are forced to define $\psi''(54) = 24 + 24$ (second block of steps). At this point we have the equality $\psi''(54) = \psi''(59)$, but we have no more space to modify $\psi''(59)$.

The following example shows that the condition of Theorem 2.3 is not necessary.

Example 2.7. Let $S = \langle 16, 17, 35, 71 \rangle$. Its Hilbert function H_R is non-decreasing; in fact it assumes the values

$$1, 4, 8, 10, 10, 11, 11, 12, 12, 13, 13, 14, 14, 15, 15, 16, \rightarrow \dots$$

For $h = 3$ we have $|D_h| = |\{52, 70, 88, 106, 142\}| = 5 > h + 1$, hence the condition of the theorem is not fulfilled; on the other hand, computing C_3 we get $\{51, 69, 87, 105, 123, 141, 159\}$; hence $|3M \setminus 4M| - |2M \setminus 3M| = 10 - 8 = 2 = |C_3| - |D_3|$.

The next result is an immediate consequence of Corollary 2.4 and Proposition 2.1.

Corollary 2.8. *If $|\{\omega_i \in \text{Ap}(S) : a_i > b_i\}| \leq 3$, then H_R is non-decreasing.*

Proof. By Proposition 2.1, we have that, if an element $s \in S$ belongs to D_h it must be of the form $s = \omega_i + kg_1 \in (h - 1)M \setminus hM$, for some ω_i such that $a_i > b_i$ and $k \in \mathbb{N}$. Furthermore, $\text{ord}(s) = h - 1$ implies that, if $k \neq k'$, then $\omega_i + k'g_1$ cannot have order $h - 1$. Thus $|D_h| \leq 3$ for every $h \in \{2, \dots, r\}$ and from Corollary 2.4 we get the thesis. \square

We notice that a particular case of the previous corollary is the well known fact that, if $g_1 = 4$ (i.e. the multiplicity of R equals 4), then H_R is non-decreasing.

The proof of the main theorem provides a computationally more efficient condition for S , in order to get that H_R is not decreasing.

Proposition 2.9. *If H_R is decreasing, then there exists an index $j \geq 2$ such that $|C_h| \geq h + 1$, for every $2 \leq h \leq j$.*

Proof. From Corollary 2.4, there exists an index $h \geq 2$ such that $|D_h| > h + 1$. Once we select $h + 1$ elements in D_h , we can use the function defined in the proof of Theorem 2.3 in order to find $h + 1$ different elements in C_h . Since each of these elements in C_h is part of a maximal representation, we can choose h different maximal representation in C_{h-1} using the same argument we used in the proof of Theorem 2.3. Again, we can select h elements in C_{h-1} and find $h - 1$ elements in C_{h-2} , and so on. \square

Remark 2.10. In particular, the index h of the thesis can be chosen as the index where the Hilbert function decreases. Hence, this could give a criterion to establish that the Hilbert function is non-decreasing without computing the cardinalities of $(h - 1)M \setminus hM$ for all the levels h . For example, we obtain that, if $|C_2| < 3$, then H_R is non-decreasing. This fact can be translated immediately in terms of the Apéry set, as follows.

Corollary 2.11. *If H_R is decreasing, then necessarily*

$$|\{\omega_i \in \text{Ap}(S) : b_i = 2\}| \geq 3.$$

Proof. By definition $C_2 = \{s \in 2M \setminus 3M : s - g_1 \notin M \setminus 2M, \}$; now, if $s - g_1 \notin M$, s necessarily belongs to the Apéry set; hence $C_2 = \{\omega_i \in \text{Ap}(S) : b_i = 2\}$. \square

Notice that Example 2.5 shows that the condition of Proposition 2.9 is not sufficient, as e.g. $|C_2| = 7$, but H_R is non-decreasing.

The next result is of some interest, since for all the known examples of decreasing H_R for numerical semigroup rings, the Hilbert function decreases at first possible step, i.e. for $h = 2$ (see, e.g. [12] and [10]). Let $e.d.(R)$ be the embedding dimension of R , i.e. the cardinality of the minimal set of generators for S .

Corollary 2.12. *If H_R is decreasing at $h = 2$, then necessarily*

$$e.d.(S) > 5.$$

Proof. Let $S = \langle g_1, g_2, \dots, g_n \rangle$ (where the generators are listed as usual in increasing order). If H_R is decreasing at $h = 2$, from Theorem 2.3, we get $|D_2| > 3$. Since $D_2 = \{g_j : g_j + g_1 \in 3M\}$ and since $g_1 + g_1$ and $g_2 + g_1$ are in $2M \setminus 3M$, the thesis follows immediately. \square

For a one-dimensional C-M local ring, if $e(R) - e.d.(R) \leq 2$, the Hilbert function is not decreasing (see e.g. [17, Theorem 4.8]); our last result shows that, in the case of numerical semigroup rings, for small embedding dimension, this difference can be increased.

Corollary 2.13. *If $e.d.(S) = 4, 5$ and $g_1 \leq 8$, then H_R is non decreasing.*

Proof. By the previous corollary H_R cannot decrease at $h = 2$. If H_R would be decreasing at $h \geq 3$, then, by Corollary 2.11, we would get $|C_2| \geq 3$ and $|C_3| \geq 4$. Now $C_3 = \{s \in 3M \setminus 4M : s - g_1 \notin 2M \setminus 3M\} = \{s \in 3M \setminus 4M : s - g_1 \in M \setminus 2M\} \cup \{s \in 3M \setminus 4M : s \in \text{Ap}(S)\}$.

If $e.d.(S) = 4$, we have $\text{Ap}(S) \supseteq \{0, g_2, g_3, g_4\} \cup C_2$; moreover $\{s \in 3M \setminus 4M : s - g_1 \in M \setminus 2M\} \subset \{g_3 + g_1, g_4 + g_1\}$. Hence, to obtain $|C_3| \geq 4$, we need at least two more elements in $\text{Ap}(S)$, so $g_1 \geq 9$.

If $e.d.(S) = 5$, we have $\text{Ap}(S) \supseteq \{0, g_2, g_3, g_4, g_5\} \cup C_2$; moreover $\{s \in 3M \setminus 4M : s - g_1 \in M \setminus 2M\} \subset \{g_3 + g_1, g_4 + g_1, g_5 + g_1\}$. Hence, to obtain $|C_3| \geq 4$, we need at least one more elements in $\text{Ap}(S)$, so, again, $g_1 \geq 9$. \square

References

- [1] F. Arslan, P. Mete, M. Sahin *Gluing and Hilbert function of monomial curves*, Proc. Amer. Math. Soc. **137** (2009), 2225-2232.
- [2] V. Barucci, D. D. E. Dobbs, M. Fontana *Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains*, Mem. Amer. Math. Soc., Vol 125, **598** (1997).
- [3] V. Barucci, R. Fröberg, *Associated graded rings of one dimensional analytically irreducible rings*, J. Algebra **304** (2006) n.1, 349-358.
- [4] T. Cortadellas Benitez, R. Jafari, S. Zarzuela Armengou *On the Apéry sets of monomial curves* Semigroup Forum **86** n.2 (2013), pp 289-320

- [5] M. D'Anna, M. Mezzasalma, V. Micale, *On the Buchsbaumness of the associated graded ring of a one-dimensional local ring*, Comm. Alg. **37** n.5 (2009), 1594 - 1603.
- [6] J. Elias, *The conjecture of Sally on the Hilbert functions for curve singularities*, J. Algebra **160** (1993), 4249.
- [7] J. Elias, J. Martnez-Borruel, *Hilbert polynomials and the intersection of ideals*, Contemp. Math. **555** (2011), 6370.
- [8] A. Garcia, *Cohen-Macaulayness of the Associated Graded Ring of a Semigroup Ring*, Comm. Algebra **10** (1982), 393-415.
- [9] S. K. Gupta, L. G. Roberts, *Cartesian squares and ordinary singularities of curves*, Comm. Algebra **11** (1983), 127182.
- [10] J. Herzog, R. Waldi, *A note on the Hilbert function of a one-dimensional Cohen-Macaulay ring*, Manuscripta Math. **16** (1975), no. 3, 251-260.
- [11] R. Jafari, S. Zarzuela Armengou *On monomial curves obtained by gluing*, Semigroup Forum **88** n.2 (2014), pp 397-416
- [12] S. Molinelli, G. Tamone *On the Hilbert function of certain rings of monomial curves*, Journal of Pure and Applied Algebra **101** (1995), 191-206.
- [13] D. G. Northcott, *On the notion of a first neighbourhood ring*, Proc. Camb. Phil. Soc. **53** (1959), 43-56.
- [14] D. G. Northcott, D. Rees, *Reduction of ideals in local rings*, Proc. Camb. Phil. Soc. **50** (1954), 145-158.
- [15] F. Orecchia, *One-dimensional local rings with reduced associated graded ring and their Hilbert functions*, Manuscripta Math. **32** (1980), 391405.
- [16] D.P. Patil, G. Tamone, *CM defect and Hilbert functions of monomial curves*, Journal of Pure and Applied Algebra **215** (2011), 1539-1551.
- [17] M. E. Rossi, *Hilbert function of Cohen-Macaulay local rings*, Commutative Algebra and its Connections to Geometry (PASI 2009), Contemporary Mathematics (2010).

Corresponding author: Marco D'Anna. e-mail address: mdanna@dmi.unict.it; fax: 0039 095 330094